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Einstein Gravitation from Scalar Nonsingular Sources Alone

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Abstract

In the light of Einstein's equations a system only containing two scalar fields is considered: One is of long range and attractive, the other is of short range and repulsive. The sources of these fields are taken to be nonsingular and spherically symmetric. All components of the energy-momentum tensor are continuous. A static solution of the equations is obtained in the weak-field approximation. The source of the gravitational field shows a finite concentration on the center of symmetry and dilutes monotonically to zero outwards. A Schwarzschild-type gravitation is found at infinity.

1. *Introduction*

One finds in the literature an increasing hope that general relativity can account for the structure and mass spectrum of the so-called elementary particles. Solutions of the Einstein equations are then looked for, in which the energy-momentum tensor is nonsingular and corresponds to quantities with a physically acceptable interpretation.

The first exact solution of the nonempty field equations is that of Schwarzschild (1916), which corresponds to an uncharged, stable, spherically symmetric static distribution of matter with uniform density; the gravitational collapse of the sphere is prevented by a pressure field. However, pressure effects are usually considered a final macroscopic result of some microscopic interactions; it seems then advisable to avoid the concept of pressure in the description of a very elementary system;

Another candidate to prevent the collapse of the system is the inertia of matter; in an Einstein cluster, for example, we have a collection of many gravitating masses in randomly oriented circular motions around a center of symmetry (Einstein, 1939; Teixeira and Som, 1974). However, besides being unstable, these clusters only postpone the solution of our problem to that of a large number of small individual massive systems.

Coulomb repulsive effects can also be introduced in order to balance the gravitational attraction; one can then consider systems with density of matter and of charge (Bonnor, 1960). One soon verifies, however, that a static equilibrium can only be maintained when these two densities bear a constant and universal ratio; this in turn implies that all static systems of that nature should have the same charge-by-mass ratio, a fact that is not observed experimentally.

Arbitrary charge-by-mass ratios were obtained (Teixeira et al., 1976) with the addition of long-range scalar fields of the type considered by Buchdahl (1959) and Wolk et al. (1975). However, one finds that two equally constructed spheres of this kind are insensitive to each other, in the sense that their mutual gravitational, electric, and scalar asymptotic effects exactly balance; this also is in disagreement with observations.

It was pointed out recently (Teixeira et al., 1975) that short-range scalar fields are very appropriate for the description of elementary systems; stable solutions were found for the static massive spheres, the constituents of which produced not only gravitation but also a repulsive short-range scalar field. The energy-momentum of the system was taken as $B_\nu{}^\mu + \rho u^\mu u_\nu$, where $B_\nu{}^\mu$ corresponds to the scalar field and ρ represents the matter density with velocity u^μ. However, that model did not fully exploit an important result of general relativity, namely, that all fields contribute to gravitation; it is then possible to obtain nonsingular solutions that do not explicitly contain matter density, but that nevertheless produce gravitational fields with physically acceptable asymptotic behavior.

In the present paper we consider the simplest system of this kind that can present stability: It contains only a diffuse source of a short-range repulsive scalar field (with one parameter, the range l) and of a long-range (zero parameter) attractive scalar field. The field equations are derived through variational principles. A class of static solutions with spherical symmetry is obtained; the solutions are regular everywhere and present the usual Schwarzschild gravitational behavior at infinity.

2. *Basic Equations*

We shall obtain our field equations from a Lagrangean density

$$\mathscr{L} = \mathscr{L}_G + \mathscr{L}_A + \mathscr{L}_B \tag{2.1}$$

$$\kappa \mathscr{L}_G = -\tfrac{1}{2}(-g)^{1/2} R, \qquad \kappa = 8\pi G/c^4 \tag{2.2}$$

$$\kappa \mathscr{L}_A = -(-g)^{1/2} A_{,\alpha} A_{,\beta} g^{\alpha\beta} + 2A\sigma_A^* \tag{2.3}$$

$$\kappa \mathscr{L}_B = (-g)^{1/2}(B_{,\alpha} B_{,\beta} g^{\alpha\beta} - B^2/l^2) + 2B\sigma_B^* \tag{2.4}$$

in these expressions R is the scalar curvature (Anderson, 1967), g is the determinant of the gravitational potentials $g_{\mu\nu}$, A is an attractive (Teixeira et al.,

1976) scalar field of long range, and B is a repulsive scalar field of short range l. An explicit dependence of $\mathscr{L}$ on the four coordinates x^μ occurs in σ_A^* and σ_B^*; these are scalar densities of weight + 1, and represent the sources of the fields A and B. Subscripted commas mean ordinary derivative.

Upon variations of the gravitational potentials $g_{\mu\nu}$ we get the Einstein equations

$$R_\nu{}^\mu = -2A^{,\mu}A_{,\nu} + 2B^{,\mu}B_{,\nu} - \delta_\nu{}^\mu B^2/l^2 \tag{2.5}$$

while the variations of the fields A and B give

$$A_{;\mu}^{;\mu} = -\sigma_A \tag{2.6}$$

$$B_{;\mu}^{;\mu} + B/l^2 = \sigma_B \tag{2.7}$$

where we introduced the scalar quantities of weight zero

$$\sigma = (-g)^{-1/2}\sigma^* \tag{2.8}$$

a semicolon means covariant derivative. The contracted Bianchi identities give, using (2.6) and (2.7),

$$\sigma_A A_{,\mu} + \sigma_B B_{,\mu} = 0 \tag{2.9}$$

We shall consider a static and spherically symmetric system, so we use the metric element

$$ds^2 = e^{2\eta}(dx^0)^2 - e^{2\alpha}\,dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\phi^2 \tag{2.10}$$

with η and α functions of r alone; with the fields A and B, and the densities σ_A and σ_B also functions of r alone, we obtain from the preceding equations

$$(\eta_{11} + 2\eta_1/r + \eta_1{}^2 - \eta_1\alpha_1)e^{-2\alpha} = B^2/l^2 \tag{2.11}$$

$$(\eta_{11} - 2\alpha_1/r + \eta_1{}^2 - \eta_1\alpha_1)e^{-2\alpha} = B^2/l^2 - 2(A_1{}^2 - B_1{}^2)e^{-2\alpha} \tag{2.12}$$

$$(\eta_1/r - \alpha_1/r + 1/r^2)e^{-2\alpha} - 1/r^2 = B^2/l^2 \tag{2.13}$$

$$r^{-2}(r^2 e^{\eta-\alpha}A_1)_1 e^{-\eta-\alpha} = \sigma_A \tag{2.14}$$

$$r^{-2}(r^2 e^{\eta-\alpha}B_1)_1 e^{-\eta-\alpha} - B/l^2 = -\sigma_B \tag{2.15}$$

with the identity

$$\sigma_A A_1 + \sigma_B B_1 = 0 \tag{2.16}$$

a subscript 1 means d/dr.

Since in the five independent equations (2.11)-(2.15) we have six functions (η, α, A, B, σ_A, σ_B), one constraint is necessary in order to get explicit solutions: It seems more natural for our purposes to consider

$$\sigma_B = f\sigma_A, \qquad f = \text{const} \tag{2.17}$$

In view of difficulty in finding exact solutions, we try an approximate method: We expand our four fields η, α, A, B and our two densities σ_A, σ_B

in integral powers of some small dimensionless parameter ϵ, to be identified later. In the lowest approximation we have taken σ_A, σ_B, A, B proportional to ϵ, and we have taken η,α proportional to ϵ^2; then (2.11)-(2.17) simplify to

$$\eta_{11} + 2\eta_1/r = B^2/l^2 \tag{2.18}$$

$$\eta_{11} - 2\alpha_1/r = B^2/l^2 - 2({A_1}^2 - {B_1}^2) \tag{2.19}$$

$$\eta_1/r - \alpha_1/r - 2\alpha/r^2 = B^2/l^2 \tag{2.20}$$

$$A_{11} + 2A_1/r = \sigma_A \tag{2.21}$$

$$B_{11} + 2B_1/r - B/l^2 = -f\sigma_A \tag{2.22}$$

$$(A_1 + fB_1)\sigma_A = 0 \tag{2.23}$$

One finds that in this order of approximation the field equations decoupled themselves; we can then use the three last equations to obtain the fields A and B, then from (2.18) we get the gravitational potential η, and finally we obtain α directly from a combination of (2.18)-(2.20):

$$\alpha = r\eta_1 - \tfrac{1}{2}r^2({A_1}^2 - {B_1}^2 + B^2/l^2) \tag{2.24}$$

3. *The Scalar Fields*

In regions where $\sigma_A \neq 0$ we get from (2.21)-(2.23)

$$B_{11} + 2B_1/r + (f^2 - 1)^{-1}B/l^2 = 0 \tag{3.1}$$

since we must have $f^2 > 1$ in order to prevent the collapse of the system (Teixeira et al., 1975), we obtain the solution regular at the origin

$$B_i = s(\nu r)^{-1} \sin \nu r, \qquad s = \text{const} \tag{3.2}$$

$$\nu^{-1} = l(f^2 - 1)^{1/2} \tag{3.3}$$

where the subscript i means internal. Then from (2.22) and (2.21) we obtain the solutions also regular at the origin

$$\sigma_A = fs\nu^2(\nu r)^{-1} \sin \nu r \tag{3.4}$$

$$A_i = -fs[(\nu r)^{-1} \sin \nu r + u], \qquad u = \text{const} \tag{3.5}$$

In regions where $\sigma_A = 0$ we obtain from (2.21) and (2.22)

$$A_e = -v/r, \qquad v = \text{const} \tag{3.6}$$

$$B_e = wr^{-1}e^{-r/l}, \qquad w = \text{const} \tag{3.7}$$

where the subscript e means external.

We now have to impose the continuity of the fields A and B, and of their radial derivatives on the boundary $r = a$ of the sphere; as a consequance of these four impositions we obtain the values of the three constants u, v, w, and also a constraint for the radius a:

$$\cot \nu a = -(f^2 - 1)^{1/2} \tag{3.8}$$

A short reflection shows that variations of sign of the densities σ_A and σ_B would induce instability in the system; one then finds from (3.4) that the only acceptable solution for νa in (3.8) is the smallest one,

$$\pi/2 < \nu a < \pi \tag{3.9}$$

which in turn implies that

$$\sin \nu a = |f|^{-1} > 0, \qquad \cos \nu a = -(f^2 - 1)^{1/2}|f|^{-1} < 0 \tag{3.10}$$

We then obtain

$$A_i = -sf[(\nu r)^{-1} \sin \nu r - \cos \nu a] \tag{3.11}$$

$$A_e = -sf \sin \nu a(1 + a/l)(\nu r)^{-1} \tag{3.12}$$

$$B_i = s(\nu r)^{-1} \sin \nu r \tag{3.13}$$

$$B_e = s \sin \nu a(\nu r)^{-1} e^{-(r-a)/l} \tag{3.14}$$

These fields A and B then present a maximum absolute value on the origin, and have a monotonic variation tending to zero as $r \to \infty$.

4. *The Gravitational Field*

We can now integrate (2.18) to obtain $\eta(r)$ continuous on the boundary a, with radial derivative also continuous. For $r \leqslant a$ we obtain

$$\eta_i = \eta(0) + \tfrac{1}{2}s^2(f^2 - 1)[(2\nu r)^{-1} \sin (2\nu r) - 1 + \ln (2\nu r) - \text{ci}(2\nu r) + C] \tag{4.1}$$

$$\eta(0) = -\tfrac{1}{2}s^2(f^2 - 1)[\ln (2\nu a) - \text{ci}(2\nu a) + C - 2f^{-2}e^{2a/l}\, \text{Ei}(-2a/l)] \tag{4.2}$$

where $C = 0.577\ldots$ is the Euler constant, and the cosine and exponential integral are

$$\text{ci}(x) = -\int_x^\infty t^{-1} \cos t\, dt, \qquad \text{Ei}(-x) = -\int_x^\infty t^{-1}e^t\, dt, \qquad x > 0 \tag{4.3}$$

for $r \geqslant a$ we obtain

$$\eta_e = -\tfrac{1}{2}s^2(f^2 - 1)r^{-1}\{a + l - f^{-2}[le^{-2r/l} + 2r\, \text{Ei}(-2r/l)]e^{2a/l}\} \tag{4.4}$$

Finally the gravitational potential $\alpha(r)$ is obtained from (2.24): We get

$$\alpha_i = -\tfrac{1}{2}s^2(f^2 - 1)(1 - \nu r \cot \nu r)(\nu r)^{-2} \sin^2 \nu r \tag{4.5}$$

$$\alpha_e = \tfrac{1}{2}s^2(f^2 - 1)r^{-1}[a + l - r^{-1}(a + l)^2 + f^{-2}l(1 + l/r)e^{-2(r-a)/l}] \tag{4.6}$$

These two expressions coincide on the boundary $r = a$.

We find that at the origin we have $\alpha(0) = 0$, while at infinity we have the usual Schwarzschild behavior

$$\eta = -\alpha = -\tfrac{1}{2}s^2(f^2 - 1)(a + l)r^{-1}, \qquad r \to \infty \tag{4.7}$$

5. *Discussion*

From (3.11) to (3.14) and from (4.1) to (4.6) we find that we can identify sf with the dimensionless parameter ϵ in terms of which we expanded our gravitational and scalar fields. Our solution is then valid for

$$|sf| \ll 1 \tag{5.1}$$

Differently from the Schwarzschild interior-exterior solution, also the derivatives α_1 and η_{11} of the gravitational potentials are continuous through the boundary $r = a$; this is a consequence of the absence of any discontinuous quantity (like density of matter in Schwarzschild solution) in the field equations (2.5).

Remembering that in Schwarzschild-type systems the mass parameter m is defined by the asymptotic behavior $\eta_e = -Gm/c^2r$, $r \to \infty$, we find from (4.7) that our system behaves gravitationally as a mass

$$m = \tfrac{1}{2}s^2(f^2 - 1)(a + l)c^2/G \tag{5.2}$$

From (2.18) we find that the contribution for this value comes solely from the square of the short-range scalar field B; so from (3.13) and (3.14) we get that the gravitational source is more concentrated in regions close to the origin, and dilutes monotonically to zero with increasing r, in a negative exponential rate.

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